CS 4814: Homework 5

due Friday 10/02, 11:59pm
(late submission until **Monday 10/05 11:59pm**)

Each question is worth 20 points.

Question 1

Let E be the following decision problem: Given a Turing machine M, a string $x \in \{0,1\}^*$, and a number t encoded in unary, decide if M on input x outputs 1 within 2^t steps. (Concretely, inputs for the problem E have the form $\langle M, x\rangle 01^t$, where $\langle M, x\rangle$ is the binary string encoding of a pair (M, x).)

Recall that $\mathrm{EXP} = \bigcup_{k\in\mathbb{N}} \mathrm{TIME}(2^{n^k})$.

Show that E is EXP-complete with respect to polynomial-time Karp reductions, i.e., show that $E \in \mathrm{EXP}$ and that $L \leq_p E$ for every problem $L \in \mathrm{EXP}$. Conclude that $\mathrm{EXP} \subseteq P^E$.

Question 2

Let Q be the following decision problem: Given a Boolean formula φ in variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$, decide if φ satisfies $\exists x \in \{0,1\}^n . \forall y \in \{0,1\}^m . \varphi(x, y)$. (Here, $\varphi(x, y)$ denotes the evaluation of the Boolean formula φ for an assignment $x \in \{0,1\}^n$ to the variables $x_1, \ldots, x_n$ and an assignment $y \in \{0,1\}^m$ to the variables $y_1, \ldots, y_m$.)

Show that $Q \in \mathrm{NP}^{\mathrm{SAT}}$.

Remark. The problem Q is believed not be in $\mathrm{P}^{\mathrm{SAT}}$. Hence, Q is a candidate to witness that $\mathrm{NP}^{\mathrm{SAT}} \neq \mathrm{P}^{\mathrm{SAT}}$. In fact, $Q \notin \mathrm{P}^{\mathrm{SAT}}$ if and only if $\mathrm{NP}^{\mathrm{SAT}} \neq \mathrm{P}^{\mathrm{SAT}}$.

Question 3

Show that deciding the satisfiability of systems of linear equations over the rationals is in NP. (As usual, we encode all numbers of the input in binary.) You may solve the case where the system is independent and has exactly one solution for partial credit.

You may use the following facts from linear algebra:

- Any matrix $A \in \mathbb{R}^{n\times n}$ with columns $a_1, \ldots, a_n \in \mathbb{R}^n$ satisfies $\det(A) \leq \|a_1\|_2 \cdots \|a_n\|_2$, where $\|x\|_2 = (x_1 + \cdots + x_n)^{1/2}$ denotes the standard Euclidean norm.
- If $A \in \mathbb{R}^{n\times n}$ is invertible, then $(A^{-1})_{j,i} = (-1)^{i+j} \cdot \det(A^{(i,j)}) / \det(A)$, where $A^{(i,j)}$ is the matrix obtained from A by removing row i and column j.
- Every matrix A contains an invertible (square) submatrix such that every column of A is a linear combination of columns of A that overlap with the submatrix.