CS 4814: Homework 3

due 09/16, 11:59pm

Each problem is worth 20 points.

Question 1

Show that if P = NP, then $\emptyset$ and $\{0,1\}^*$ are the only languages that are not NP-hard.

Question 2

Show that if a language $L \subseteq \{0,1\}^*$ satisfies $L \leq_p$ 3 SAT then $L \in$ NP. Conclude using the Cook-Levin theorem that the set of NP-complete languages is the equivalence class of 3 SAT induced by $\leq_p$ (that is, the equivalence class under the relation $=_p$ where $L_1 =_p L_2$ if and only if $L_1 \leq_p L_2$ and $L_2 \leq_p L_1$).

Question 3

A 2-CNF (conjunctive normal form) formula with variables $x_1, \ldots, x_n$ is a set of clauses of the form $C_i = \ell_{i,1} \vee \ell_{i,2}$, where $\ell_{i,1}, \ell_{i,2} \in \{x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n\}$ are literals.

We consider the following decision problems about such formulas:

2 SAT: Given a 2-CNF formula φ, decide if there exists an assignment to the variables of φ that satisfies all clauses of φ.

MAX 2 SAT: Given a 2-CNF formula φ and a number k, decide if there exists an assignment to the variables of φ that satisfies at least k clauses of φ.

a. Show that 2 SAT has a polynomial-time algorithm.

b. Show that there exists a 2-CNF formula φ with variables x_1, x_2, x_3, y with the following two properties:

   i. For every assignment to x_1, x_2, x_3 that satisfies the clause $x_1 \vee x_2 \vee x_3$, there exists an assignment to y such that 8 clauses of φ are satisfied, and no assignment to y satisfies more than 8 clauses of φ.

   ii. For every assignment to x_1, x_2, x_3 that does not satisfy the clause $x_1 \vee x_2 \vee x_3$, no assignment to y exists such that more than 7 clauses of φ are satisfied.

c. Use the formula φ from 3.b as a gadget to show that 3 SAT $\leq_p$ MAX 2 SAT. (If you haven't solved part 3.b (yet), you can assume a formula as described in 3.b. Your NP hardness reduction for MAX 2 SAT and its analysis should only use the properties of φ stated in 3.b.)