Each problem is worth 20 points.

**Question 1**

Show that if $P = \text{NP}$, then $\emptyset$ and $\{0, 1\}^*$ are the only languages that are not NP-hard.

**Question 2**

Show that if a language $L \subseteq \{0, 1\}^*$ satisfies $L \leq_p \text{3SAT}$ then $L \in \text{NP}$. Conclude using the Cook-Levin theorem that the set of NP-complete languages is the equivalence class of 3SAT induced by $\leq_p$ (that is, the equivalence class under the relation $=_p$ where $L_1 =_p L_2$ if and only if $L_1 \leq_p L_2$ and $L_2 \leq_p L_1$).

**Question 3**

A 2-CNF (conjunctive normal form) formula with variables $x_1, \ldots, x_n$ is a set of clauses of the form $C_i = \ell_{i,1} \lor \ell_{i,2}$, where $\ell_{i,1}, \ell_{i,2} \in \{x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n\}$ are literals.

We consider the following decision problems about such formulas:

**2 SAT:** Given a 2-CNF formula $\varphi$, decide if there exists an assignment to the variables of $\varphi$ that satisfies all clauses of $\varphi$.

**Max 2 SAT:** Given a 2-CNF formula $\varphi$ and a number $k$, decide if there exists an assignment to the variables of $\varphi$ that satisfies at least $k$ clauses of $\varphi$.

a. Show that 2 SAT has a polynomial-time algorithm.

b. Show that there exists a 2-CNF formula $\varphi$ with variables $x_1, x_2, x_3, y$ with the following two properties:

   i. For every assignment to $x_1, x_2, x_3$ that satisfies the clause $x_1 \lor x_2 \lor x_3$, there exists an assignment to $y$ such that 8 clauses of $\varphi$ are satisfied, and no assignment to $y$ satisfies more than 8 clauses of $\varphi$.

   ii. For every assignment to $x_1, x_2, x_3$ that does not satisfy the clause $x_1 \lor x_2 \lor x_3$, no assignment to $y$ exists such that more than 7 clauses of $\varphi$ are satisfied.

c. Use the formula $\varphi$ from 3.b as a gadget to show that 3 SAT $\leq_p$ Max 2 SAT. (If you haven’t solved part 3.b (yet), you can assume a formula as described in 3.b. Your NP hardness reduction for Max 2 SAT and its analysis should only use the properties of $\varphi$ stated in 3.b.)